



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 50

Black Scholes Model

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EXAMPLE 1

- Consider a stock that pays no dividends, has a volatility (σ) of 30% per annum, and provides an expected return (μ) of 15% per annum with continuous compounding and has the current price of 1,000. What is the probability of a price increase of 54 or more in one week. (Assume that “one week” qualifies as an “infinitesimally small” time period so that we can use the usual stock price model for infinitesimally small time periods. Also assume 1 week = 0.0192 year).



VOLATILITY	0.3	
EXPECTED RETURN	0.15	
CURRENT PRICE	1000	
TIME	0.0192	YEARS
IN THE INFINITESIMAL MODEL		
CHANGE IN STOCK PRICE (dS) IS NORMALLY DISTRIBUTED		
WITH MEAN $\mu S dt$	2.88	
SD $\sigma S \sqrt{dt}$	41.56921938	
DESIRED PRICE INCREASE	54	
CORRESPONDING Z VALUE	1.229756073	
REQUIRED PROB	0.1093	

EXAMPLE 2

- The price of a stock follows the lognormal distribution with $\mu = 17\%$ p.a. and σ (volatility) = 20% p.a. The current stock price is 100. What is the probability of the price increasing to more than 149 at the end of two years from today?

EXPECTED RETURN μ		0.1700
VOLATILITY σ		0.2000
CURRENT STOCK PRICE $S(0)$		100.0000
PROJECTED STOCK PRICE $S(T)$		149.0000
TIME HORIZON T		2.0000
EXPECTED VALUE OF $\ln S(T)$	$\ln S(0) + [\mu - 0.5\sigma^2]T$	4.9052
SD OF $\ln S(T)$	$\sigma\sqrt{T}$	0.2828
$\ln S(T)$	$\ln 149$	5.0039
Z VALUE		0.3492
$P(Z > 0.3492)$		0.3632

ANALYSIS OF STOCK PRICE MODELS						
BASIC DATA						
INSTANTANEOUS RETURN						0.15
STD DEVIATION PER ANNUM (Vol)						0.3
CURRENT STOCK PRICE						1000
EXPECTED STOCK PRICE						1054
TIME FRAME						0.0192308



SHORT TERM MODEL						
MEAN						1002.8846
SD						41.602515
Z VALUE						1.2286609
LONG TERM MODEL						
LN (S)						6.9077553
MEAN OF LN(S)						6.9097745
SD OF LN(S)						0.0416025
LN(S(T))						6.9603477
Z VALUE						1.2156289
						0.013032

BASIC DATA						
INSTANTANEOUS RETURN						0.15
STD DEVIATION PER ANNUM (Vol)						0.3
CURRENT STOCK PRICE						1000
EXPECTED STOCK PRICE						1054
TIME FRAME						1
SHORT TERM MODEL						
MEAN						1150
SD						300
Z VALUE						-0.32
LONG TERM MODEL						
LN (S)						6.90776
MEAN OF LN(S)						7.01276
SD OF LN(S)						0.3
LN(S(T))						6.96035
Z VALUE						-0.17469
						-0.14531



EXAMPLE

- A stock's price follow a lognormal distribution with $\mu = 12\%$ p.a. and $\sigma = 30\%$ p.a. Calculate the probability that the stock's price at the end of nine months will be greater than its expected price at that time.

μ	0.12
σ	0.3
TIME	0.75
$S(0)$	1000
$S(T)$	1094.174284
$\ln(S(0))$	6.907755279
MEAN LN PRICE AFTER NINE MONTHS	6.964005279
SD	0.259807621
NOW, $X = \ln S(T)$	6.997755279
CORRESPONDING Z VARIATE VALUE (z)	0.129903811
$P(Z < z)$	0.5517
$P(Z > z)$	0.4483

BLACK SCHOLES MODEL



NOTATION

$$\begin{aligned} &\text{If } Z \xrightarrow{\text{distribution}} \mathbf{N}(0,1) \\ &\text{then } P(Z \leq \alpha) = \Phi(\alpha) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} d\omega \mathbf{exp}\left(-\frac{\omega^2}{2}\right) \end{aligned}$$

Pr ove $1 - \Phi(\alpha) = \Phi(-\alpha)$

$$\begin{aligned} I &= 1 - \Phi(\alpha) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} d\omega \exp\left(-\frac{\omega^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} d\omega \exp\left(-\frac{\omega^2}{2}\right). \text{ Set } \omega = -\theta, \quad d\omega = -d\theta \\ \text{so that } I &= -\frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{-\infty} d\theta \exp\left(-\frac{\theta^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\alpha} d\theta \exp\left(-\frac{\theta^2}{2}\right) = \Phi(-\alpha) \end{aligned}$$

ASSUMPTIONS

- The stock price follows the lognormal process with constant mean return and volatility.
- The short selling of securities with full use of proceeds is permitted.
- There are no transactions costs or taxes. All securities are perfectly divisible.



- There are no dividends during the life of the derivative.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.
- The risk-free rate of interest, r , is constant and the same for all maturities.

**BLACK
SCHOLES
PDE**

Consider a derivative $G \equiv f(S, t)$ (1)

(1) The stock S follows the SDE :

$$dS = \mu S dt + \sigma S dW \quad (2)$$

(2) Ito's Lemma :

$$dG = \left(a \cdot \frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial x^2} \right) dt + b \cdot \frac{\partial G}{\partial x} dW \quad (3)$$

$$\text{where } dx = a dt + b dW \quad (4)$$

(3) Set $G \equiv f(S, t)$; $x \equiv S$; $a \equiv \mu S$; $b \equiv \sigma S$ (5)

$$df = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW \quad (6)$$

(4) Construct riskless portfolio Π
→ 1 unit of derivative

→ $-\frac{\partial f}{\partial S}$ units of stock

$$\Pi = f - \frac{\partial f}{\partial S} S \quad (7)$$

(5) Change in value of Π due to small change dS in S in time dt

$$d\Pi = df - \frac{\partial f}{\partial S} dS \quad (8)$$

Using eq (2) & eq (6) in eq (8)

$$d\Pi = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW - \frac{\partial C}{\partial S} (\mu S dt + \sigma S dW) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \quad (9)$$

Now, eq (9) does not contain any stochastic term. Hence, the portfolio Π is riskfree and will generate riskfree return r over the interval dt so that

$$\frac{1}{\Pi} \frac{d\Pi}{dt} = r \text{ or } d\Pi = r\Pi dt = r \left(f - \frac{\partial f}{\partial S} S \right) dt \quad (10)$$

$$d\Pi = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{t}} + \frac{1}{2} \boldsymbol{\sigma}^2 \mathbf{S}^2 \frac{\partial^2 \mathbf{f}}{\partial \mathbf{S}^2} \right) d\mathbf{t} \quad (9)$$

$$d\Pi = \mathbf{r} \left(\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{S}} \mathbf{S} \right) d\mathbf{t} \quad (10)$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} + \frac{1}{2} \boldsymbol{\sigma}^2 \mathbf{S}^2 \frac{\partial^2 \mathbf{f}}{\partial \mathbf{S}^2} + \mathbf{r} \mathbf{S} \frac{\partial \mathbf{f}}{\partial \mathbf{S}} - \mathbf{r} \mathbf{f} = \mathbf{0} \quad (11)$$

BOUNDARY CONDITIONS

For a European call option :

$$f(S_T, T) = c(S_T, T) = \max(S_T - K, 0)$$

For a European put option :

$$f(S_T, T) = p(S_T, T) = \max(K - S_T, 0)$$



BS SOLUTIONS FOR CALLS & PUTS

$$c = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$p = Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON DIVIDEND PAYING STOCK

$$c = (S_0 - D_0) \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$p = Ke^{-rT} \Phi(-d_2) - (S_0 - D_0) \Phi(-d_1)$$

$$d_1 = \frac{\ln((S_0 - D_0)/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln((S_0 - D_0)/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON INDICES

$$c = S_0 e^{-qT} \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

$$p = K e^{-rT} \Phi(-d_2) - S_0 e^{-qT} \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON CURRENCY

$$c = S_0 e^{-r_f T} \Phi(d_1) - K e^{-r T} \Phi(d_2)$$

$$p = K e^{-r T} \Phi(-d_2) - S_0 e^{-r_f T} \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r - r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - r_f - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$